# DIFFERENTIAL TOPOLOGY FINAL - NOVEMBER 7 2014

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Each question is of the marks indicated adding up to a total of 50 marks.

1a. Show that if $\omega$ is a closed 1-form on $S^1$, then (5)

$$\omega \text{ is exact} \Leftrightarrow \int_{S^1} \omega = 0$$

1b. Show that there exists closed forms on $S^1$ which are *not* exact. Conclude that $H^1_{deRham}(S^1, \mathbb{R}) \simeq \mathbb{R}$. (5)

2a. Suppose $X = \partial W$, $W$ compact, and $f : X \to Y$ is a smooth map. Let $\omega$ be a *closed* $k$-form on $Y$, $k = \dim(X)$. Prove that if $f$ extends to all of $W$, then (5)

$$\int_X f^*(\omega) = 0$$

2b. Use part (a) to show that if $f_0, f_1 : X \to Y$ are homotopic maps from $X$ a compact, boundaryless manifold of dimension $k$, then for all closed forms $\omega$ on $Y$, (5)

$$\int_X f_0^*(\omega) = \int_X f_1^*(\omega)$$

2c. Conclude that if $X$ is simply connected, then $\int_\gamma \omega = 0$ for all closed 1-forms $\omega$ and all closed curves $\gamma$. (5).

3. Prove that $S^2$ and the torus $T^2$ are not diffeomorphic. (5)

4. Show that $GL_n(\mathbb{R})$, the group of $n \times n$ invertible matrices, forms a manifold. What is its dimension? (5)

5. Let $f : X \to X$ be a map with a fixed point $x$ – that is, $f(x) = x$. If $+1$ is **not** an eigenvalue of $df_x$ then $x$ is called a *Lefschetz fixed point* and $f$ is called *Lefschetz* if all of its fixed points are Lefschetz. Show that if $X$ is compact and $f$ is Lefschetz, then $f$ has only finitely many fixed points. (10)

6. Suppose $f : X \to Y$ has $\deg_2(f) \neq 0$, where $\deg_2(f)$ is the mod 2 degree. Prove that $f$ is onto. Recall that for $\deg_2$ to be defined, there have to be some conditions on $X$ and $Y$. (5)